Open Problems for Convex Polytopes I’d Love to See Solved

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The $g$-Conjecture for Spheres

The $g$-conjecture for simplicial spheres asserts that the set of $f$-vectors of simplicial $(d - 1)$-dimensional spheres is identical to the set of $f$-vectors of simplicial $d$-polytopes.

Let $K$ be a $d$-dimensional simplicial complex. Define the $h$-vector of $K$, $h(K) = (h_0, h_1, \ldots, h_d)$ by the relation

$$\sum_{i=0}^{d} h_i x^{d-i} = \sum_{i=0}^{d} f_{i-1} (x - 1)^{d-i}.$$

Let $g_i = h_i - h_{i-1}$. 

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Convex polytopes
The $g$-conjecture II

The $g$-conjecture: A vector $h$ is the $h$-vector of a simplicial $(d - 1)$-dimensional sphere iff

(i) $h_i = h_{d - i},$

(ii) $g_{i - 1} \leq g_i$, $i = 0, 1, \ldots [d/2] - 1,$

(iii) $g_i$ is bounded above by a certain nonlinear functions (described by Macaulay) of $g_{i - 1}$.

Conditions (ii) and (iii) together asserts that $(g_0, g_1, \ldots, g_{[d/2]})$ is an $M$-sequence, namely it counts according to degrees the monomials in a graded ideal of monomials.
A quick review

- The lower bound theorem
- The upper bound theorem
- The generalized lower bound conjecture
- The g-conjecture
- The upper bound theorem for spheres
- Sufficiency for the $g$-Conjecture for polytopes
- Necessity for the $g$-theorem for polytopes
- A new proof for the necessity part
- The generalized lower bound theorem

Additional proofs; wider contexts
The $g$-conjecture: Basic Problems

Prove the $g$-conjecture for spheres.

Understand the equality cases for the nonlinear inequalities

Extend to larger classes of simplicial complexes (manifolds, pseudomanifolds,...)
The Polynomial Hirsch Conjecture

The Hirsch Conjecture: The diameter of the graph of a $d$-polytope with $n$ facets is at most $n - d$.

The Polynomial Hirsch Conjecture: The diameter of the graph of a $d$-polytope with $n$ facets is at most polynomial in $n$ and $d$. 
A quick review

- The simplex algorithm for linear programming
- Hirsch Conjecture
- Klee-Walkup counterexamples for the unbounded case
- Linear upper bounds for fixed dimension
- Connection with vertex-decomposability
- Quasi polynomial upper bounds; massive abstraction
- Examples meeting the Hirsch bound
- Santos’ counterexample to the Hirsch conjecture
Developments and Further Problems

Hähnle’s conjecture: The diameter for strongly connected family of degree $d$ monomials with $n$ variables is at most $d(n - 1)$.

Theorem: (Adiprasito and Benedetti): The Hirsch bound holds for flag simplicial spheres

Theorem (de Lorea and Klee): There are examples for non-$k$-vertex decomposable polytopes.
Two questions by Barany

Is it the case that for every $d$-polytope

$$f_k \geq \min\{f_0, f_{d-1}\}$$

Is the number of maximal chains of faces of a $d$-polytope bounded above by a constant (depending on $d$) times the number of faces of all dimensions? In other words, is

$$f_{\{0,1,2,...,(d-1)\}} \leq C_d (f_1 + f_1 + \cdots + f_{d-1})?$$
Flag vectors of general polytopes

- Stanley’s thesis: enumerating chains in graded posets
- Bayer-Billera: The space of flag vectors has dimension $F_d$ (the $d$th Fibonacci number.
- The toric $g$-vector; GLBT for toric $g$-vectors of rational polytopes
- The CD-index
- FLAGTOOL
- GLBT for toric $g$-vectors of general polytopes (Karu)
Flag vectors of general polytopes

**Problem:** Understand linear inequalities for flag vectors and find consequences.

**Conjecture:** The toric $g$-vector is an M-sequence

**Problem:** Understand polytopes with $g_k(P) = 0$. (The case $k = 2$ is already fascinating.)

**Problem:** Does the toric GLBC extend to polyhedral spheres? (The lower bound inequality $g_1 \leq g_2$ for polyhedral 3-spheres is already fascinating.)
Counting Spheres and Polytopes

**Problem** What is the number of combinatorial types of triangulations of $d$-spheres with $n$ facets? Is it exponential in $n$?

**Problem:** What is the number of (combinatorial types of) triangulations of $d$-spheres with $n$ vertices?
Four-polytopes

The fatness of a 4-polytope (or a 3-D polyhedral sphere) is defined by

$$\frac{f_1(P) + f_2(P)}{f_0(P) + f_3(P)}.$$ 

Problem (Ziegler): are there arbitrary fat 4-polytopes?
Two universality conjectures

Conjecture (Perles): Every simplicial polytope (sphere) is the quotient of a neighborly polytope (sphere).

Conjecture: Every polytope is combinatorially isomorphic to a subpolytope of a stacked polytope.
UBCs for unusual faces: two questions

**Conjecture** (Ziegler): The maximum number of non-quadrangles 2-faces for a simple 4-polytopes with $n$ facets is at most linear in $n$.

**Question** (Nevo): What is the maximum number of facets of a $d$-polytope with $n$ vertices which are not simplices. (Especially for 4-polytopes).
Puzzles

The puzzle of a pure $d$-dimensional simplicial complex is its dual graph (vertices are facets; two facets are incident if they share a $(d - 1)$-face).

**Conjecture:** Simplicial spheres are determined by their puzzle

**Conjecture:** If $K$ and $L$ are $d$-dimensional complexes with the same puzzle and if $K$ is Cohen-Macaulay then $f_i(K) \geq f_i(L)$ for every $i$. If $L$ is not Cohen Macaulay then $f_i(K) > f_i(L)$ for some $i$. 
More topics:

Shellability, PL, Pachner moves, etc

Linear Programming

Constructions and examples

Around Steinitz theorem

Finer numerical and algebraic invariants of polytopes
Facets, non-facets and low dimensional faces

**Problem:** Is there a 4-polytope all whose facets are icosahedra?

**Conjecture:** For every $k$ there is $d(k)$ such that every $d$-polytope, $d \geq d(k)$ has a $k$-face which is a simplex or a cube.

**Problem:** Are there 5-simple 5-simplicial polytopes?
The generalized upper bound conjecture

**Conjecture:** Let $P$ be a $d$-polytope, if for some $i$,

$$f_i(P) \leq f_i(C(d, m)),$$

then for every $j > i$,

$$f_j(P) \leq f_j(C(d, m)).$$

The same applies for polyhedral spheres, and for subcomplexes of polyhedral spheres and more general objects. A more detailed Kruskal-Katona type conjecture is available. Linear inequalities follow, including a positive answer to Barany’s first question.
Special classes of polytopes

- Centrally-symmetric
- Cubical;
- Flag;
- Completely balanced
- Zonotopes

- The $3^d$ Conjecture
- How neighborly can CS polytopes be
- Adin’s GLBT for cubical polytopes
- The Charney-Davis conjecture and finer conjectures about Gal’s $\gamma$-vector.
Conclusion